How old is the (Pythagorean) theorem?

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| Year levelStrand(s)Lesson lengthCD code | Year 8Measurement / Number60 mins[AC9M8M06,](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-7_year-8_year-9_year-10/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8M06&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) [AC9M8N01](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-7_year-8_year-9_year-10/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8N01&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) |
| Lesson summary | This is an introductory lesson in a series of lessons on Pythagoras theorem. Students learn about Pythagoras’ theorem and its application in calculating lengths in right-angled triangles. They explore the statement of the theorem, some of its history, a simple geometric proof and visual demonstrations of the theorem. Students apply the theorem to the two types of problem calculating the length of the third side of a right-angled triangle, given the lengths of the other two sides.  |
| Learning intention | * We are learning to state and explain Pythagoras’ theorem.
* We are learning to understand some of its history.
* We can apply the theorem to find lengths in right-angled triangles.
* We know how to determine when a triangle is right-angled**.**
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| Success criteria | By the end of this lesson, students can:* state the theorem and draw and label the corresponding diagram for specific examples
* find the length of the longest side (hypotenuse) of a right-angled triangle, given the lengths of the other two sides (the ‘legs’)
* find the length of a shorter side of a right-angled triangle given the lengths of the longest side and the length of the other side
* determine when a triangle is a right-angled triangle or not given the lengths of its three sides.
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| Why are we learning about this? | Pythagoras’ theorem is a fundamental theorem of plane geometry that connects lengths of sides of a triangle with right-angles and defines length and distance in plane geometry. The theorem has practical applications in design, building and surveying, where calculating and constructing diagonal distances that intersect perpendicular lines is important. |
| **Prerequisite student knowledge and language** | Prior to this lesson, it is assumed that students:* have knowledge of triangle inequality (that the sum of the lengths of any two sides must be greater than the length of the remaining side)
* are familiar with the spatial terms: point, line segment, side, right-angle, triangle and square and their representation and labelling in diagrams
* are familiar with the measurement terms: 90 degrees, unit, cm, square cm, length, area
* are familiar with the number terms and notations for integer, square and square root.

The following terms will need to be introduced in the context of the theorem: leg, opposite, hypotenuse. |
| **Resources** | * Teacher’s slides (PowerPoint)
* How old is the theorem worksheet (Word)
* Toothpicks, icy-pole sticks or other manipulatives that can be used as unit length line segments
* 1-cm-grid graph paper, scientific calculator
* Laptops / access to computers, dynamic geometric software
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Curriculum information

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| Achievement standard | Students use Pythagoras’ theorem to solve measurement problems involving unknown lengths of right-angled triangles. By the end of Year 8, students recognise irrational numbers and terminating or recurring decimals.  |
| Content description(s) | Students recognise irrational numbers in applied contexts, including square roots and *π.*[AC9M8N01](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8N01&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick)Students use Pythagoras’ theorem to solve problems involving the side lengths of right-angled triangles.[AC9M8M06](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8M06&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) |
| General capabilitiesCross-curriculum priority | **General capabilities**Numeracy* Understanding geometric properties ([Level 7](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NM&sub-element-index=0&sub-element-code=NMUGP&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))
* Understanding units of measurement ([Level 10](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NM&sub-element-index=1&sub-element-code=NMUuM&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))
* Multiplicative strategies ([Level 9](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NN&sub-element-index=0&sub-element-code=NNMuS&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))

Critical and Creative Thinking* Interpret concepts and problems ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=CCT&element-code=CCTANA&sub-element-index=0&sub-element-code=CCTANAA&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))
* consider alternatives ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=CCT&element-code=CCTGEN&sub-element-index=0&sub-element-code=CCTGENB&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))

Digital Literacy* Select and operate tools ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum.html/learning-areas/mathematics/year-8/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=DL&element-code=DLMO&sub-element-index=0&sub-element-code=DLMOC&elaboration-code=AC9M8N01_E2&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))
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| Areas of challenge | * Some students may not yet realise that while the triangle inequality applies for all triangles, Pythagoras’ theorem only applies to right-angled triangles; the careful use of counterexamples of triangles with side lengths {*a*, *b*, *c*} that satisfy the inequality but not the equation for Pythagoras’ theorem will help illustrate the difference.
* Some students may have little experience dealing with square numbers and square roots and applying these operations on a calculator; completing a table of values of square numbers from 1 to 30 and calculating square roots of numbers like 87, 243 and 456 with a calculator and noting which numbers they lie between will be helpful for familiarisation.
* Some students may not yet understand that the square root of a sum/difference of two numbers is not the same as the sum/difference of the square roots of the two numbers; a selection of suitably chosen counterexamples will be of assistance, for example, $\sqrt{9+16}=\sqrt{25}=5$ but $\sqrt{9 }+\sqrt{16} = 3 + 4 = 7$.
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| Strategies | [Mathematics investigation](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/mathematics-investigation/)[Questioning](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/questioning/)[Explicit teaching](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/explicit-teaching/)[Concrete, Representational, Abstract (CRA)](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/concrete-representational-abstract-cra/)[Worked examples](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/worked-examples/) |

Lesson structure

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| Learning hook15 mins | **General notes** This lesson combines a historical narrative as context and stimulus with exploration supported by questioning and discussion. The teacher sets the scene for the proof exploration by outlining the principles of the approach to developing a proof. Students are encouraged to look for a result that is in the form of an equationconnecting the relationship to three sides {*a*, *b*, *c*} of a right-angled triangle. Use physical manipulatives (such as toothpicks or icy-pole sticks) or a dynamic geometric software (GeoGebra-related resources, for example, [Triangle Inequality Exploration – GeoGebra](https://www.geogebra.org/m/JT5UmZ8w)) to experiment and explore different combinations of side lengths, which are multiples of a given unit length, to form specified types of triangles, including a right-angled triangle. Students work individually or in small groups, and opportunities to briefly present their solutions and explain their reasoning are given.**Learning hook**To begin, use slides 2 and 3 to inform students about the lesson and what they will be learning and achieving. Distribute the 12 toothpicks or unit-length sticks to each student and go to slide 4. Pose the following scenario below. Use 12 unit-length sticks to form the 3 sides of a triangle. All 12 sticks must be used. What side lengths are required to make:* + an equilateral triangle?
	+ an isosceles triangle?
	+ a right-angled triangle?

Let students explore the task. Slides 5–7 give the solutions. Discuss as a class or have students communicate their findings. **Differentiation** (enable): * What is an equilateral triangle? If there are 12 unit-length sticks, how many must be on each side?
* What is an isosceles triangle? How can you make up 12 by adding two equal numbers and another number?

Some related questions/points for discussion are: * Can a triangle be formed with one side 6 units in length? What happens in this case?
* Are there any other possible solutions to these problems when there are 12 unit-length sticks?

**Differentiation** (extend): What solutions are possible if you can use 30 unit-length sticks?**Digging into history**From here students are exposed to the historical contexts behind Pythagorean theorem. Slide 8 shows ancient texts from the Babylonians and ancient Egyptians. Comment on the fact that the {3, 4, 5} triangle is a right-angled triangle was well known to ancient civilisations from around 4000 years ago and was used as a measuring or surveying tool to make a right-angle. Further historical points of discussion are included in the teaching notes on this slide.* State the ‘triangle inequality rule’ that the sum of two lengths of any triangle is greater than the length of the remaining side. (This concept is covered in the lesson: Making triangles.)
* Expand this by adding that although Pythagoras’ theorem is geometrically related to the inequality rule, it is different because it is an equation that describes a relationship specific to **only to right-angled triangles** and more specifically between the areas of squares based on the sides of a right-angled triangle.
* Why is this important? Because this relationship because it gives exact calculations of side lengths in right-angled triangles, which has practical applications in various fields.
* For instance, theorem enables the design of buildings with precise angles and dimensions. Similarly, in land surveying, the theorem can determine distances and therefore land boundaries without estimating.
* Historically it meant that the theorem helped to develop architecture, design, engineering, navigation, and those sciences that rely on precise measurements.
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| Explore35 mins | **This is where the students come in!** Go to slide 9 for students to consider the diagram for the {3, 4, 5} right-angled triangle and have students note the areas of the squares and calculate that 32 + 42 = 52 that is 9 + 16 = 25.* Prompt: If the theorem is like a universal-game cheat code for right-angle triangles, (that no matter how large or small a right-angle triangle is the side lengths or angles can be determined every time), how can we demonstrate that this outcome works for **any** right-angled triangle?

Explicitly outline the following arguments for a proof strategy.* If two things are equal to a third thing, then they are equal to each other.
* Start with *any* right-angled triangle with side lengths {*a*, *b*, *c*) where *c* > *a*, *b.* Discuss the various names/terms for the sides: the **hypotenuse** is the longest side, and the side opposite the right angle, the two shorter sides a, b are sometimes called the ‘legs’ of the triangle or its **base** and **altitude** / **height**.
* Show how these are labelled on a diagram with the right-angled triangle.
* Identify the squares on each side of the triangle, *a*, *b*, *c*, and note their areas are respectively *a*2, *b*2 and *c*2.

Comment that the proof students are going to explore involves no algebra (other than writing the equation for Pythagoras’ theorem) and is based on visually recognising areas that must be the same (equal) for shapes that fit exactly within an identical larger square using two different arrangements.**Explore** (slides 10 and 11)* Download and distribute ‘How old is the theorem?’ worksheet found in the What you need section of the online lesson. You may wish to pre-cut the triangles before the class begins; or students can work on their computers using Microsoft Word shapes. (In a Word document, go to Insert Tab > Shapes to create a right-angled triangle, make 3 copies of it, and arrange these in two different ways to fit into an *a* + *b* side length square.)
* Have students work in small groups with four identical right-angled triangles to arrange them in **two different ways**so that they fit exactly into a larger square frame with side length *a* + *b*.
* Ask students to explain how these two diagrams, with the same total area, show that the sum of the areas of the squares on the two smaller sides (the two ‘legs’) is equal to the area of the square on the larger side (the hypotenuse).
* Students could be asked to demonstrate their group’s reasoning using a whiteboard, or projecting images from a computer screen.
* Students will either have observed the following themselves or can be prompted to note that there are exactly two ways to do this.
* Fitting in exactly by arrangement four {*a*, *b*, *c*} right-angled triangles, a square with side length *a*, and area *a*2, and a square with side length *b*, and area *b*2 (slide 12).
* Fitting in exactly by arrangement four {*a*, *b*, *c*} right-angled triangles, a square with side length *c*, and area *c*2.(slide 13)

The following question can be used to lead to a statement of the theorem:* The two big *a* + *b* square areas are the same, and each contains four {*a*, *b*, *c*} right-angled triangles, so what does this tell us about the remaining areas?
* This leads to the conclusion that the area of the ‘*a*’ square plus the area of the ‘*b*’ square is equal to the area of the ‘*c*’ square; that is, *a*2 + *b*2 = *c*2.

Reinforce this by showing, and discussing with the class, the following two animations: * [Animated gif version of SVG of rearrangement proof of Pythagorean theorem - Pythagorean theorem - Wikipedia](https://en.wikipedia.org/wiki/Pythagorean_theorem#/media/File:Animated_gif_version_of_SVG_of_rearrangement_proof_of_Pythagorean_theorem.gif)
* [Pythag anim - Pythagorean theorem - Wikipedia](https://en.wikipedia.org/wiki/Pythagorean_theorem#/media/File:Pythag_anim.gif)

Explicitly teach how to solve problems using the theorem.This will involve the processes of:* substitution of the two known lengths into the formula and evaluating
* re-arranging to make the square of the unknown side length the subject of the equation
* using a calculator to evaluate the square root and find the unknown side length to a suitable accuracy.

Lead a class discussion to identify the two types of problems that can be solved using the theorem, and illustrate the process of solving them * Find the missing values for each of the following right-angled triangles {8, 20, *c*} (slide 14) and {10, *b*, 17} (slide 15).
* Students then generate a similar problem for each type. Go through the [worked examples](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/worked-examples/) together in a class discussion.
* Students work in small groups to generate and solve similar problems of their own and present solutions on the board and explain reasoning.
* Iteration: The triangles could be drawn to scale on 1-cm-grid graph paper, then the required lengths can be measured and compared with the values calculated from the theorem.
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| Summary and reflection10 mins | Display the pair of diagrams and re-cap how they show that *a*2 + *b*2 = *c*2 (slide 16). Re-cap an example of the application of the theorem to each of the two cases of a missing side in a right-angled triangle. Consider using additional examples or refer to previous examples. Alternatively, a flow chart with examples could be developed to summarise the process. |
| Assessment | The following opportunity for assessment is suggested below.The exit ticket is found on slide 17 in the downloadable Teacher’s slides. Students work through the following questions in their exercise books for you to collect at the end of the lesson.Question 1: Can {7, 11, 14} be the side lengths of a right-angled triangle?Question 2: If {a, 12, 18} are the side lengths of a right-angled triangle, what is the value of *a*?Question 3: If {8, 8, c} are the side lengths of a right-angled triangle, what is the value of c?Answers: No, as 49 + 121 ≠ 196; 13.42 (correct to two decimal places); 11.31 (correct to two decimal places). |