Pythagorean triples

|  |  |
| --- | --- |
| Year level  Strand(s)  Lesson length  CD code | Year 8  Measurement, Number  60 mins  [AC9M8M06,](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-7_year-8_year-9_year-10/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8M06&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) [AC9M8N01](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-7_year-8_year-9_year-10/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8N01&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) |
| Lesson summary | This lesson is one in a series of lessons on Pythagoras theorem. The purpose of this lesson is to have students undertake a mathematical exploration to find Pythagorean triples, that is, sets of positive integers {*a*, *b*, *c*} such that *a*2 + *b*2 = *c*2. Students apply trial and error to identify sets of positive integers {*a*, *b*, *c*} that are Pythagorean triples using a spreadsheet. They apply scaling to develop new triples from a known triple, and use a formula (Euclid’s formula) to generate Pythagorean triples. They then take a systematic approach to explore if there are any emerging patterns. This lesson is an example of an exploration within mathematics itself rather than an application of mathematics to contexts involving practical problems or other learning areas. |
| Learning intention | * We are learning to identify when a set of positive integers {*a*, *b*, *c*} forms a Pythagorean triple. * We will use scaling to create new Pythagorean triples from a known Pythagorean triple. * We are learning to apply Euclid’s formula to generate new Pythagorean triples. * We will take a systematic approach. |
| Success criteria | By the end of this lesson students can:   * verify if a given set of positive integers {*a*, *b*, *c*} forms a Pythagorean triple * generate new Pythagorean triples from a given Pythagorean triple * use a formula to generate new Pythagorean triples * take a systematic approach to mathematical investigations. |
| Why are we learning about this? | The exploration of Pythagorean triples provides the opportunity to develop the mathematical investigation process, explore patterns and build fluency in working with square numbers and algebraic expressions. This develops some of the approaches used in working mathematically in mathematical investigation (trial and error, systematic exploration, generalisation).  Pythagorean triples are not just theoretical constructs but can be applied in real-world situations, like architecture, construction, and navigation, where right angles are crucial. |
| **Prerequisite student knowledge and language** | Prior to this lesson, it is expected that students have:   * knowledge of Pythagoras’ theorem and the formula *a*2 + *b*2 = *c*2 * knowledge of the number terms and notations for integer, square and square root * familiarity with squares of small integers and the concept of a perfect square * familiarity with the use of a scientific calculator. |
| Resources | * Teacher’s slides (PowerPoint) * Testing triples and Euclid's formula spreadsheet (Excel) * Scientific calculator, laptops / access to a computer, geometric dynamic software |

Current information

|  |  |
| --- | --- |
| Achievement standard | By the end of Year 8, students recognise irrational numbers and terminating or recurring decimals. They use Pythagoras’ theorem to solve measurement problems involving unknown lengths of right-angle triangles. |
| Content description(s) | * Students recognise irrational numbers in applied contexts, including square roots and π*.*[AC9M8N01](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8N01&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) * Students use Pythagoras’ theorem to solve problems involving the side lengths of right-angled triangles**.** [AC9M8M06](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8/content-description?subject-identifier=MATMATY8&content-description-code=AC9M8M06&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick) |
| General capabilities  Cross-curriculum priority | **General capabilities**  Numeracy   * Understanding geometric properties ([Level 7](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NM&sub-element-index=0&sub-element-code=NMUGP&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick)) * Understanding units of measurement ([Level 10](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NM&sub-element-index=1&sub-element-code=NMUuM&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick)) * Multiplicative strategies ([Level 9](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8M06&general-capability-code=N&element-code=NN&sub-element-index=0&sub-element-code=NNMuS&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))   Critical and creative thinking   * Interpret concepts and problems ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=CCT&element-code=CCTANA&sub-element-index=0&sub-element-code=CCTANAA&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick)) * Consider alternatives ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum/learning-areas/mathematics/year-8_year-7_year-9_year-10/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=CCT&element-code=CCTGEN&sub-element-index=0&sub-element-code=CCTGENB&load-extra-subject=MATMATY7_MATMATY8_MATMATY9_MATMATY10&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&achievement-standard=04f8219e-c3b7-4ecf-a612-701f06ecf8f7&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick))   Digital literacy   * Select and operate tools ([Level 5](https://v9.australiancurriculum.edu.au/f-10-curriculum.html/learning-areas/mathematics/year-8/general-capability-snapshot?subject-identifier=MATMATY8&content-description-code=AC9M8N01&general-capability-code=DL&element-code=DLMO&sub-element-index=0&sub-element-code=DLMOC&elaboration-code=AC9M8N01_E2&detailed-content-descriptions=0&hide-ccp=0&hide-gc=0&side-by-side=1&strands-start-index=0&subjects-start-index=0&view=quick)) |
| Areas of challenge | * Some students lack knowledge of square numbers. This can be addressed by having students create a reference table of squares on numbers from 1 to 20, and practising calculating squares of larger numbers using a calculator or another tool such as a spreadsheet. * Some students may have difficulty interpreting and applying algebraic expressions. This can be addressed by writing out a list of written instructions for applying a formula and practising these. Students will have previously been exposed to and worked with applications of Pythagoras’ theorem to calculate lengths, distances, as well as lengths in the coordinate plane. This will have developed the ability to interpret and apply formulas involving squaring, addition and subtraction (sums and differences). |
| Strategies | [Mathematics investigation](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/mathematics-investigation/)  [Questioning](https://www.mathematicshub.edu.au/plan-teach-and-assess/teaching/teaching-strategies/questioning/) |

Lesson structure

|  |  |
| --- | --- |
| Learning hook  10 mins | Note that this lesson combines trial and error exploration, verification by substitution and calculation, systematic exploration, and the interpretation and application of algebraic expressions involving square terms. Note that calculators or spreadsheets can be used at any stage to support calculations, and tables used to record results. Students are not expected to derive the formula provided to generate new triples; the purpose is to systematically explore its use. A spreadsheet is a useful tool for this purpose, and an Excel file with two specifically designed sheets is available to download. Use the teacher’s slides to complement the teaching of this lesson.  **Learning hook**  Begin the lesson with the learning intention and success criteria shown on slides 2 and 3 of the teacher’s slides.  Then follow with a brief introduction on what students know: ‘We know that the {3, 4, 5} triangle is a right-angled triangle that satisfies Pythagoras’ theorem *a*2 + *b*2 = *c*2 because 32 = 9, 42 = 16, 52 = 25 and 9+ 16 = 25. Ancient civilisations had discovered a range of Pythagorean triples and had methods for generating them from around 4000 years ago. Let's see if we can discover some of these for ourselves.’  Show slide 4. Explain that a Pythagorean triple is anyset of 3 positive integers *{a*, *b*, *c*} written in order from smallest to largest, such that *a*2 + *b*2 = *c*2. They are of interest because they offer exact, integer solutions, avoiding the complexities and approximations often encountered in practical problems involving decimals or irrational numbers. Pose the following:  See if you can find two other Pythagorean triples involving one- or two-digit integers.  Use a calculator to square numbers and a table to record your work.  **Differentiation** (enable): Is {6, 8, 10} a Pythagorean triple? How do you know? Is {5, 6, 7} a Pythagorean triple? How do you know?  **Differentiation** (extend): What is the value of *c* so that {5, 12, *c*} is a Pythagorean triple? There are two triples of the form {*a*, *b*, 25}, what are they?  Ask students to share some of the triples they have found and record several of these on the board.  Explain that Pythagorean triples are not just theoretical constructs but can be applied in real-world situations, like architecture, construction, and navigation, where right angles are crucial. |
| Explore  40 mins | Slides 5–8 give a general idea of the investigation, teacher’s notes, and then break down the investigation into its constituent parts. Notes are given for differentiation.  **General teaching notes**   * As students work through Part 1 and Part 2 of the investigation, they will need to decide how to: systematically explore identifying primitive triples and multiples of these triples; record and represent these using a table; summarise and report their findings. * Students will record results in a table of their won devising or use the preprepared Excel Spreadsheet tool: Testing triples and Euclid's formula, which can be found in the What you need section. Note the spreadsheet includes two tabs: Testing triples for Part 1, and Euclid’s formula for Part 2. This prompts them to work in a systematic manner and in turn allows patterns to emerge and connections to be made. * Discuss that as this lesson is an investigation, there is no set of ‘correct’ outcomes, rather it’s about the process of working mathematically to obtain some results, organising and representing these, discussing approaches, interpreting and communicating findings. This will involve identifying sets of numbers {*a*, *b*, *c*} that do not form Pythagorean triples as well as those that do. Each part of the exploration should take around 10–15 minutes. * Observe progress and approaches to facilitate discussion in the summary and reflection stage of the lesson. * Assure students that you will guide them through the investigation; slide 6 gives background knowledge that will help students make connections with the task, mathematical process and language.   **Exploration**  To begin, introduce the main investigation for the lesson shown on slide 5. Explain that the investigation will be divided into two parts.  Students will be working in groups or pairs to explore triples and multiples. They will record results systematically and they will also report back a summary of their findings.  **Background information** Explain there are two types of Pythagorean triples:  **First type:**   * Triples that **do not** have a common factor are called ‘primitive’triples, for example, {3, 4, 5} or {5, 12, 13}.   **Second type:**   * Triples that **do** have a common factor. * For example, {9, 12, 15} is a multiple of {3, 4, 5} by a common factor of 3. * {10, 24, 26} is a multiple of {5, 12, 13} by a common factor of 2. * Point out that these triples are multiples of ‘primitive’ triples. Have students state some of the other triple multiples of {3, 4, 5} and discuss as a class to check understanding.   **Part 1** (slide 7):  Instruct students to use trial and error to identify some of the primitive triples. Ask them to find the 16 primitive triples involving one-digit or two-digit numbers. The calculations can be done using the Excel spreadsheet or the resource: [Pythagorean Triple – GeoGebra](https://www.geogebra.org/m/DjY6ZH3N).  **Differentiation** **part 1** (enable): use the Excel spreadsheet and prompt students as they explore. Prompts include:   * There are five triples that involve a one-digit number. * Look at a table of squares from 1 to 30 and try some combinations for {*a*, *b*, *c*} that look like they might work when considering the size of their square numbers. * There are two different triples that involve 12. * For each primitive triple, record all its multiples involving one- or two-digit numbers, multiplying by a factor of 2, 3, 4 … until the largest is just smaller than 100.   **Differentiation part 1** (extend):  Try the following prompts.   * What is the largest number used in a triple that involves only one- or two-digit numbers? * There are sixteen primitive triples involving a three-digit number up to 200. * What is the largest number used in a triple that involves only two- or three -digit numbers less than 200?   **Part 2** (slide 8):  Introduce Euclid’s formula and explain that it provides a systematic method to generate Pythagorean triples, including integer values in the hundreds, thousands or larger.   * Choose any two positive integers *m* and *n* where *m* > *n.* * Calculate:  *a* = *m*2 – *n*2*b* = 2 *× m* × *n c* = *m*2 + *n*2*.* * Check *a2* + *b2* = *c*2.   Work through an example on the board: *m* = 2, *n* = 1 and show how this corresponds to the well-known {3, 4, 5} triple: *a* = 4 – 1 = 3, *b* = 2 × 2 × 1 = 4, *c* = 4 + 1 = 5.  Then have students work through the case where *m* = 3, *n* =2, to generate the triple {5, 12, 13} and check that 52 + 122 = 132 (25 + 144 = 169).  Students then use Euclid’s formula to explore using the formula to find triples involving 2-digit and 3-digit numbers. Encourage students to notice any patterns emerging. For example, ask students to keep the value of *m* = 5 and iteratively change the value of *n* to 1, 2 and to observe the corresponding values *a, b, c,* and notice if there is there an emerging pattern. Ask, can you use it to predict the triple when *m* = 5 and *n* = 4?  **Differentiation** **part 2** (enable):   * Try combinations where *m* is one bigger than *n.* * What values of *m* and *n* generate the triple {9, 40, 41}.   **Differentiation** **part 2** (extend):   * What happens for different combinations of *m* and *n* being odd or even? * Note that Euclid’s formula does not always generate primitive triples, for example, when *m* = 5 and *n* = 3 it generates the triple {16, 30, 34} for which the corresponding primitive triple is {8, 15, 17}. This is found by dividing the values in {16, 30, 34} by their common factor of 2. * The resources: [Pythagorean Triples – GeoGebra - 2](https://www.geogebra.org/m/p6vTEbmq) or [Pythagorean Triples – GeoGebra - 3](https://www.geogebra.org/m/ajCnkcF8) can be used to check working. |
| Summary and reflection  10 mins | Arrange for several students to briefly present their findings from Part 1 and Part 2 of the investigation.  Collate these on the board using a table to capture different sets of triples and take the opportunity to ask the whole class if there are any others found. The GeoGebra resource [Pythagorean triples map – GeoGebra](https://www.geogebra.org/m/US7hextt) provides a nice visual summary and could be shown after student results have been collated.  Draw out the conclusion that the formula means there are an infinite number of triples, as one can choose an infinite number of combinations of *m* and *n*. This could be done via a prompting question such as, *‘*how many triples do you think there could be …?’  For further reading, go to the primitive Pythagorean triples with 1-, 2- or 3-digit are listed here: [Pythagorean triple - Wikipedia](https://en.wikipedia.org/wiki/Pythagorean_triple). |
| Assessment | The following assessment opportunity is suggested in the exit ticket found below and in the teacher’s slides on slide 9.  Question 1: which one of the following is a Pythagorean triple: {4, 6, 8}, {6, 8, 10}, {8, 10, 12}?  Question 2: what is the value of *b* for {12, *b*, 37} to be a Pythagorean triple?  Answers: {6, 8, 10}, *b* = 35 |